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# The stability of long, steady, two-dimensional salt fingers

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In this paper we study the stability of long, steady, two-dimensional salt fingers. It is already known that salt fingers carrying a large enough density flux are unstable to long-wavelength internal-wave perturbations. Stern (1969) studied the mechanism of this, the collective instability, and it was studied in more detail by Holyer (1981). We extend the earlier work to include perturbations of all wavelengths, as well as long-wavelength perturbations. By applying the methods of Floquet theory to the periodic salt fingers, the growth rates of perturbations are found. For both heat–salt and salt–sugar systems the collective instability, which can be recognized by its frequency of oscillation, does not have the largest growth rate. There is a new, non-oscillatory instability, which, according to linear theory, grows faster than the collective instability. We study the instabilities that arise by using a combination of analytical and numerical methods. Further work will be necessary in order to assess the importance of these instabilities in different physical situations and to examine their development as their amplitude increases.

## 1. Introduction

Over the past two decades the importance of double diffusion to transport processes in the ocean has been recognized. The earliest description of a double-diffusive process occurs in the classic paper of Stommel, Arons & Blanchard (1956), where 'the perpetual salt fountain' is referred to as 'an oceanographic curiosity'. Stern (1960) gives the first theoretical explanation of double diffusion, and since then the subject has been growing. Initially all applications of double diffusion were to oceanography. More recently it has been realized that double diffusion is an important phenomenon in many other areas, such as vulcanology, the melting of icebergs, solar ponds, crystal growth, polymer solutions and stellar evolution (Huppert & Turner 1981; Chen & Johnson 1984).

There are two double-diffusive mechanisms. The essential feature that is needed for double diffusion to occur in a fluid is the presence of two components which diffuse at different rates. If the faster-diffusing component, which we shall call $T$, makes a stable contribution to the density gradient and the slower-diffusing component, which we shall call $S$, makes an unstable contribution, then salt-fingering can occur. If the faster-diffusing component makes an unstable contribution to the density gradient and the slower-diffusing component makes a stable contribution, then an oscillatory instability occurs. In this paper only the first of these two mechanisms is considered, namely salt fingering.

Salt fingers have been observed in the Mediterranean outflow by Williams (1974) and in the North Atlantic Current by Schmitt & Georgi (1982). The observations show that salt fingers are confined to thin regions, about 20 cm thick, and are separated by convecting regions several metres thick. This suggests that, although very long salt-finger motions satisfy the equations of motion, the salt fingers themselves are unstable and eventually breakdown into layers.

Stern (1969) investigated the stability of long, steady, two-dimensional salt fingers to long-wavelength internal-wave perturbations. He shows that if the fluxes through the fingers are large enough then the fingers are unstable. This instability is known as the collective instability of salt fingers. Holyer (1981) studied this instability more rigorously and showed that salt fingers are unstable to long-wavelength internal-wave perturbations if

$$\frac{\beta F_S - \alpha F_T}{\nu(\alpha T_z - \beta S_z)} > \frac{1}{3}, \tag{1.1}$$

where $F_T$ and $F_S$ are the heat and salt fluxes of the salt fingers, $\nu$ is the kinematic viscosity of the fluid, and $T_z$ and $S_z$ are the heat and salt gradients. An experiment was performed by Stern & Turner (1969) on a field of long salt fingers in a salt–sugar system. This experiment confirmed the physical existence of the collective instability and was in reasonable agreement with the theory. Experiments by Linden (1973) and Schmitt (1979) on the thin salt-finger layer that exists between two well-mixed layers of different temperature and salinity showed that for the heat–salt system the stability parameter, given by (1.1), had values ranging from 0.2 to 1.9 in the salt-finger layer. Lambert & Demenkow (1972) carried out a similar experiment using salt and sugar. They found that the stability parameter had a value of approximately 0.002. A more careful experiment by Griffiths & Ruddick (1980) has the effect of increasing the parameter by an order of magnitude, but that still leaves it two orders of magnitude less than the theoretical prediction of (1.1). This leads one to explore the possibility that there is another instability present that determines the vertical extent of the salt-finger region.

In this paper we study the stability of long, steady, two-dimensional salt fingers in an infinite fluid to two-dimensional perturbations of all wavelengths. It is expected that there will be no qualitative changes if three-dimensional motions are also considered, and it is intended that later work will look at the stability of fingers with a square cross-section in order to determine the quantitative changes. From the present study we can find, for any field of salt fingers, the perturbation with the maximum growth rate, and its corresponding wavelength. We find that the collective instability does not always have the largest growth rate, but that other instabilities can grow faster. If the Prandtl number is large, then the collective instability grows fastest. In other circumstances the fastest-growing instability is non-oscillatory.

The methods we use are similar to those used by Beaumont (1981), who used Floquet theory to examine the stability of spatially periodic homogeneous flows. It is already known that internal waves are linearly unstable for all amplitudes (Drazin 1977). We show here that salt fingers are always linearly unstable, however small the salt-finger flux. For small fluxes the growth rates are small, and so the instabilities are not observed experimentally. The collective instability appears when the fluxes through the fingers are larger than some critical value.

We find a new, non-oscillatory instability that grows fastest when the perturbation has the same period in the horizontal as the salt fingers. In §4 we find an analytic expression for its growth rate that is valid when the perturbation has a long vertical wavelength. The dimensionless growth rate $\lambda$ for a salt finger with maximum vertical velocity $\hat{W}$ is

$$\lambda = -\tfrac{1}{2}\sigma m^2 + (\tfrac{1}{2}m^2 \hat{W}^2 + \tfrac{1}{4}\sigma^2 m^4)^{\frac{1}{2}}, \tag{1.2}$$

where $\sigma$ is the Prandtl number and $m$ the vertical wavelength of the perturbation. The velocity $\hat{W}$ is related to the fluxes by

$$\hat{W}^2 = \frac{2(\alpha F_T - \tau\beta F_S)}{\kappa_T(\alpha T_z - \beta S_z)}, \tag{1.3}$$

where $\kappa_T$ is the thermal diffusivity and $\tau$ is the ratio $\kappa_S/\kappa_T$. In §5 we examine this instability as $m$ varies, and show that there is a value of $m$ for which it has a maximum growth rate. This instability often has growth rates larger than those of the collective instability, which can be recognized because the perturbations oscillate at close to the internal-wave frequency. It is possible that in some experiments it is this instability, rather than the collective instability, that is being observed.

This paper presents an initial study of the linear stability of salt fingers to perturbations of all wavelengths. For long-wavelength perturbations at large Prandtl number we verify the results of Holyer (1981) for the collective instability. We find the wavelength of the perturbations that gives the maximum growth rate for the collective instability. We also investigate other instabilities that are present. In §4 we examine the instabilities that are present for long wavelength perturbations, in particular the non-oscillatory instability. In §5 we present detailed numerical results for the growth rates of the instabilities.

## 2. The salt fingers

We consider motion in an unbounded region of incompressible fluid, which has a stable linear temperature gradient $T_z$ and an unstable linear salinity gradient $S_z$, with the overall density gradient statically stable. Coordinates $(x, z)$ are chosen with $x$ horizontal and $z$ vertically upwards. Only two-dimensional motions are considered in this paper. A stream function $\psi$ can then be defined by

$$u = -\frac{\partial\psi}{\partial z}, \quad w = \frac{\partial\psi}{\partial x}, \tag{2.1}$$

where $u$ is the horizontal velocity and $w$ is the vertical velocity. The temperature field $T'$ and the salinity field $S'$ are given by

$$T' = T_z z + T(x, z, t), \quad S' = S_z z + S(x, z, t). \tag{2.2a, b}$$

The density field is given by

$$\rho = \rho_0(1 - (\alpha T_z - \beta S_z)z - (\alpha T - \beta S)), \tag{2.3}$$

where $\alpha$ and $\beta$ are the coefficients of expansion for heat and salt, with $\alpha$ and $\beta$ positive. In order that the density gradient is statically stable, we require that

$$\alpha T_z > \beta S_z > 0. \tag{2.4}$$

This also ensures that the temperature gradient is stable and the salinity gradient unstable.

The two-dimensional equations of motion are then

$$\frac{\partial}{\partial t}\nabla^2\psi + \mathrm{J}(\psi, \nabla^2\psi) = \frac{\partial}{\partial x}(g(\alpha T - \beta S)) + \nu\nabla^4\psi, \tag{2.5a}$$

$$\frac{\partial}{\partial t}T + \mathrm{J}(\psi, T) + T_z\frac{\partial\psi}{\partial x} = \kappa_T\nabla^2 T, \tag{2.5b}$$

$$\frac{\partial}{\partial t}S + \mathrm{J}(\psi, S) + S_z\frac{\partial \psi}{\partial x} = \kappa_S \nabla^2 S, \tag{2.5c}$$

where
$$\mathrm{J}(\psi, \phi) = \frac{\partial \psi}{\partial x}\frac{\partial \phi}{\partial z} - \frac{\partial \psi}{\partial z}\frac{\partial \phi}{\partial x}$$

is the Jacobian and $\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial z^2$. The thermal diffusivity is $\kappa_T$ and the saline diffusivity is $\kappa_S$. Following Holyer (1981), we non-dimensionalize using a length-scale $l$, a timescale $l^2/\kappa_T$, a temperature scale $lT_z$ and a salinity scale $lS_z$. The non-dimensional equations are then

$$\frac{1}{\sigma}\left[\frac{\partial}{\partial t}\nabla^2\psi + \mathrm{J}(\psi, \nabla^2\psi)\right] = R_T\frac{\partial T}{\partial x} - R_S\frac{\partial S}{\partial x} + \nabla^4\psi, \tag{2.6a}$$

$$\frac{\partial}{\partial t}T + \mathrm{J}(\psi, T) + \frac{\partial \psi}{\partial x} = \nabla^2 T, \tag{2.6b}$$

$$\frac{\partial}{\partial t}S + \mathrm{J}(\psi, S) + \frac{\partial \psi}{\partial x} = \tau\nabla^2 S, \tag{2.6c}$$

where
$$\sigma = \frac{\nu}{\kappa_T}, \quad \tau = \frac{\kappa_S}{\kappa_T}, \quad R_T = \frac{\alpha g T_z l^4}{\nu\kappa_T}, \quad R_S = \frac{\beta g S_z l^4}{\nu \kappa_T}. \tag{2.7}$$

We look for a steady $z$-independent solution to (2.6), representing the motion in the salt fingers. Such a solution is

$$\psi = -\hat{W}\cos x, \quad T = \hat{T}\sin x, \quad S = \hat{S}\sin x, \tag{2.8}$$

where
$$\hat{T} = -\hat{W}, \quad \hat{S} = -\frac{\hat{W}}{\tau}, \quad \hat{W} = R_T\hat{T} - R_S\hat{S}. \tag{2.9}$$

Equations (2.9) imply that

$$R_S = \tau(1 - R_T). \tag{2.10}$$

In dimensional terms (2.10) determines the lengthscale, $l$, by

$$l^4 = \frac{\nu}{\beta g S_z/\kappa_S - \alpha g T_z/\kappa_T}. \tag{2.11}$$

The amount of heat transported downwards by the salt fingers is

$$F_T = -\overline{\frac{\partial \psi}{\partial x}T}\frac{\kappa_T}{l}lT_z$$

where $\overline{(\ )}$ denotes a horizontal average. So

$$F_T = -\kappa_T T_z \hat{T}\hat{W}\overline{\sin^2 x} = \tfrac{1}{2}\kappa_T T_z \hat{W}^2. \tag{2.12}$$

Similarly the amount of salt transported downwards is given by

$$F_S = \tfrac{1}{2}\kappa_T S_z \frac{\hat{W}^2}{\tau}. \tag{2.13}$$

We then define the flux ratio (Turner 1979) by

$$\gamma = \frac{\alpha F_T}{\beta F_S}. \tag{2.14}$$

Using (2.12) and (2.13), we obtain

$$\gamma = \frac{\alpha T_z}{\kappa_T}\frac{\kappa_S}{\beta S_z}. \tag{2.15}$$

Then, by the definitions of $R_T$ and $R_S$ and by (2.10), we find

$$\gamma = \frac{\tau R_T}{R_S} = \frac{R_T}{1+R_T}. \tag{2.16}$$

We can find bounds on the size of $\gamma$. Equation (2.4) implies that

$$R_T > R_S. \tag{2.17}$$

Hence, by (2.10), we find
$$R_T > \frac{\tau}{1-\tau}. \tag{2.18}$$

Thus, using (2.16), we have
$$\gamma > \tau. \tag{2.19}$$

Also, since $R_T > 0$, we have $\gamma < 1$, so

$$1 > \gamma > \tau. \tag{2.20}$$

The independent dimensionless parameters that we use here to specify the salt fingers are $\sigma, \tau, \gamma$ and $\hat{W}$. Any other dimensionless parameters can be expressed in terms of these. In §3 we consider the stability of the fingers.

Some comment needs to be made about the relevance of studying steady fingers. There are also growing salt-finger solutions to (2.6), with

$$\psi = -\hat{W} e^{\lambda t} \cos kx, \quad T = \hat{T} e^{\lambda t} \sin kx, \quad S = \hat{S} e^{\lambda t} \sin kx, \tag{2.21}$$

where

$$\hat{W} = -\hat{T}(\lambda + k^2), \quad \hat{W} = -\hat{S}(\lambda + \tau k^2), \quad \frac{\hat{W}}{\sigma}(\lambda + \sigma k^2) = R_T \hat{T} - R_S \hat{S}. \tag{2.22}$$

Equations (2.22) then give an equation for $\lambda$:

$$(\lambda + \sigma k^2)(\lambda + k^2)(\lambda + \tau k^2) + \sigma(R_T(\lambda + \tau k^2) - R_S(\lambda + k^2)) = 0. \tag{2.23}$$

If when investigating the stability of steady fingers we find growth rates that are large compared with $\lambda$, then the fingers will be almost steady, i.e. quasi-steady, for the time it takes for the perturbations to grow. So, provided we find growth rates for the stability of steady salt fingers that are large compared with the largest salt-finger growth rates obtained from (2.23), it is appropriate to study the stability of steady fingers.

## 3. The perturbations

We now perturb the salt fingers by putting

$$\left.\begin{aligned} \psi &= -\hat{W}\cos x + \psi(x,z,t), \\ T &= \hat{T}\sin x + T(x,z,t), \\ S &= \hat{S}\sin x + S(x,z,t). \end{aligned}\right\} \tag{3.1}$$

Substituting into (2.6) and linearizing in the perturbation quantities yields

$$\left(\frac{1}{\sigma}\frac{\partial}{\partial t} - \nabla^2\right)\nabla^2\psi - R_T\frac{\partial T}{\partial x} + R_S\frac{\partial S}{\partial x} = -\frac{\hat{W}}{\sigma}\sin x\left(\frac{\partial}{\partial z}\nabla^2 + \frac{\partial}{\partial z}\right)\psi, \tag{3.2a}$$

$$\left(\frac{\partial}{\partial t} - \nabla^2\right)T + \frac{\partial \psi}{\partial x} = -\hat{W}\sin x\frac{\partial T}{\partial z} - \hat{W}\cos x\frac{\partial \psi}{\partial z}, \tag{3.2b}$$

$$\left(\frac{\partial}{\partial t} - \tau\nabla^2\right)S + \frac{\partial \psi}{\partial x} = -\hat{W}\sin x\frac{\partial S}{\partial z} - \frac{\hat{W}}{\tau}\cos x\frac{\partial \psi}{\partial z}. \tag{3.2c}$$

Holyer (1981) obtained these equations and used them to study the collective instability of salt fingers. A scale separation was used on the above equations to separate the small-scale motion of the fingers from the large-scale motion of the internal-wave perturbations. We now show how to solve the set of equations (3.2) exactly, by using Floquet theory. We find solutions to (3.2) for all wavelength perturbations. We also obtain new asymptotic results for long-wavelength perturbations.

Since the coefficients of (3.2) are independent of $z$ and $t$, we can find solutions with $\psi$, $T$ and $S$ proportional to $\exp(\mathrm{i}mz+\mathrm{i}\omega t)$. The coefficients in (3.2) are periodic in $x$, with period $2\pi$, and hence the solutions can be written in the Floquet form

$$\begin{pmatrix} \psi \\ T \\ S \end{pmatrix} = \exp(\mathrm{i}kx+\mathrm{i}mz+\mathrm{i}\omega t) \sum_{n=-\infty}^{\infty} \begin{pmatrix} \psi_n \\ -\mathrm{i}T_n \\ -\mathrm{i}S_n \end{pmatrix} \exp(\mathrm{i}nx). \tag{3.3}$$

Substituting this form of solution into (3.2) gives the following set of equations, for each $n$:

$$\mathrm{i}\omega\psi_n = -\sigma K_n^2\,\psi_n - \frac{(n+k)\,\sigma R_T}{K_n^2} T_n + \frac{(n+k)\,\sigma R_S}{K_n^2} S_n + \frac{m\hat{W}}{2K_n^2}\,[(K_{n+1}^2-1)\,\psi_{n+1} - (K_{n-1}^2-1)\,\psi_{n-1}], \tag{3.4a}$$

$$\mathrm{i}\omega T_n = -K_n^2\,T_n + (n+k)\,\psi_n + \tfrac{1}{2}m\hat{W}(T_{n+1}-T_{n-1}) + \tfrac{1}{2}m\hat{W}(\psi_{n+1}+\psi_{n-1}), \tag{3.4b}$$

$$\mathrm{i}\omega S_n = -\tau K_n^2\,S_n + (n+k)\,\psi_n + \tfrac{1}{2}m\hat{W}\,(S_{n+1}-S_{n-1}) + \frac{m\hat{W}}{2\tau}\,(\psi_{n+1}+\psi_{n-1}), \tag{3.4c}$$

where $K_n^2 = (n+k)^2+m^2$. This set of equations is invariant under the transformations $k \to k+1$ and $k \to -k$, so only values of $k$ between 0 and $\frac{1}{2}$ need to be considered.

Equations (3.4) can be written in the matrix form

$$\mathrm{i}\omega \boldsymbol{x} = \boldsymbol{A}\boldsymbol{x} \tag{3.5}$$

where $\boldsymbol{x}$ is the column vector $(\ldots\psi_{n-1}\,T_{n-1}\,S_{n-1}\,\psi_n\,T_n\,S_n\ldots)^{\mathrm{T}}$; $\boldsymbol{A}$ is the infinite matrix obtained from (3.4) and is a function of $\sigma, \tau, \gamma$, $\hat{W}$, $k$ and $m$. Equation (3.5) specifies the infinite-matrix problem for the eigenvalues $\mathrm{i}\omega = \lambda$ and the eigenvectors $\boldsymbol{x}$. This equation cannot, in general, be solved analytically. In order to solve it for all $\sigma, \tau, \gamma$, $\hat{W}$, $k$ and $m$, we truncate the system (3.5) at some order $N$, so that $\psi_n$, $T_n$ and $S_n$ are all assumed to be zero for $|n| > N$. The number $N$ is chosen to be large enough that if $N$ is increased by 1 then the eigenvalue with the largest real part does not change by more than $10^{-4}$. The manipulations to solve (3.5) were then programmed in Fortran on Bristol University's Honeywell system using routines from the NAG library. If, on solving (3.5), we find that the eigenvalue with the largest real part has positive real part then the flow in unstable, and if it has a negative real part the flow is stable.

In §5 we shall discuss the results obtained from this numerical scheme. First, however, we shall look at how this work can be compared with that of Holyer (1981) and we also obtain some new analytical results for long-wavelength perturbations.

## 4. Long-wavelength perturbations

Holyer (1981) studied the linear stability problem (3.2) by considering perturbations that varied over a long horizontal scale and by separating the long-lengthscale motion from the short-lengthscale motions which are forced by the presence of the salt fingers. The analysis performed in Holyer (1981) is formally equivalent to truncating (3.5) at order one, i.e. $N = 1$. The long-lengthscale perturbations are proportional to $\exp(\mathrm{i}kx+\mathrm{i}mz+\mathrm{i}\omega t)$, and the short-scale motions forced by the fingers are proportional to $\exp(\mathrm{i}(k\pm 1)x+\mathrm{i}mz+\mathrm{i}\omega t)$. This drastic truncation of (3.5) gives solutions to the full problem (3.2) only if $\psi_n$, $T_n$ and $S_n$ are negligible for $|n| \geqslant 2$, i.e. if they are small compared with $\psi_0, \psi_1, \psi_{-1}$ etc. This is only the case if $\mu^2 = K_0^2 = k^2+m^2 \ll 1$. By solving (3.5) numerically we can compare solutions obtained from the first-order truncation with the numerical solutions that can be found for larger values of $k$ and $m$. Holyer (1981) only looked for solutions with $\mathrm{i}\omega$ of zeroth order in $\mu^2$. In this section we obtain some new solutions that were not found in the earlier paper. In §5 we verify that the solutions that are found from the truncation can also be found from numerical solutions to the full equations (3.5).

Performing the first-order truncation on (3.5) gives

$$\left(\frac{\mathrm{i}\omega}{\sigma}+K_1^2\right)K_1^2\psi_1+(k+1)(R_T T_1-R_S S_1)=\frac{m\hat{W}}{2\sigma}(1-\mu^2)\psi_0, \tag{4.1$a$}$$

$$(\mathrm{i}\omega+K_1^2)T_1-(k+1)\psi_1=\frac{m\hat{W}}{2}(\psi_0-T_0), \tag{4.1$b$}$$

$$(\mathrm{i}\omega+\tau K_1^2)S_1-(k+1)\psi_1=\frac{m\hat{W}}{2}\left(\frac{\psi_0}{\tau}-S_0\right), \tag{4.1$c$}$$

$$\left(\frac{\mathrm{i}\omega}{\sigma}+K_{-1}^2\right)K_{-1}^2\psi_{-1}+(k-1)(R_T T_{-1}-R_S S_{-1})=-\frac{m\hat{W}}{2\sigma}(1-\mu^2)\psi_0, \tag{4.2$a$}$$

$$(\mathrm{i}\omega+K_{-1}^2)T_{-1}-(k-1)\psi_{-1}=\frac{m\hat{W}}{2}(\psi_0+T_0), \tag{4.2$b$}$$

$$(\mathrm{i}\omega+\tau K_{-1}^2)S_{-1}-(k-1)\psi_{-1}=\frac{m\hat{W}}{2}\left(\frac{\psi_0}{\tau}+S_0\right), \tag{4.2$c$}$$

$$\left(\frac{\mathrm{i}\omega}{\sigma}+\mu^2\right)\mu^2\psi_0+k(R_T T_0-R_S S_0)=\frac{m\hat{W}}{2\sigma}[(K_1^2-1)\psi_1-(K_{-1}^2-1)\psi_{-1}], \tag{4.3$a$}$$

$$(\mathrm{i}\omega+\mu^2)T_0-k\psi_0=\frac{m\hat{W}}{2}[(T_1-T_{-1})+(\psi_1+\psi_{-1})], \tag{4.3$b$}$$

$$(\mathrm{i}\omega+\tau\mu^2)S_0-k\psi_0=\frac{m\hat{W}}{2}\left[(S_1-S_{-1})+\frac{1}{\tau}(\psi_1+\psi_{-1})\right]. \tag{4.3$c$}$$

This truncation will give results close to the exact results for $\mu^2 \ll 1$.

In Holyer (1981) solutions to (4.1)–(4.3) were found by assuming $\mathrm{i}\omega = O(1)$, $k/m = O(1)$ and $\sigma \gg 1$. The collective instability of salt fingers, first studied by Stern (1969), was then obtained. This gave instability if

$$\frac{\hat{W}^2}{2\sigma(R_T-R_S)}=\frac{\beta F_S-\alpha F_T}{\nu(\alpha T_z-\beta S_z)}>\frac{\mu^2}{3m^2} \tag{4.4}$$

and $$\frac{\omega^2}{\sigma} = \frac{k^2}{\mu^2}(R_T - R_S) + O(\mu^2). \tag{4.5}$$

Hence for the collective instability the perturbation is a growing oscillatory instability, oscillating at close to the internal-wave frequency.

We now look for additional solutions to (4.1)–(4.3). The Prandtl number $\sigma$ is no longer assumed to be large. First we note that when $m = 0$ we can solve (3.5) exactly, for salt fingers of any amplitude $\hat{W}$. In this case the salt fingers already present do not interact at all with the growth of any perturbations. (This is most easily seen by setting $m = 0$ in (3.4).) Then, for each $n$, we obtain the equation

$$\lambda^3 + \lambda^2 K_n^2(\sigma+\tau+1) + \lambda\left(K_n^4(\sigma\tau+\sigma+\tau) + \frac{\sigma(n+k)^2}{K_n^2}(R_T - R_S)\right) + \sigma\tau(K_n^6 - (n+k)^2) = 0, \tag{4.6}$$

where $\lambda = \mathrm{i}\omega$. This is the standard equation for the growth of salt fingers in an infinite region of constant gradients (Turner 1979). Equation (4.6) can be shown to have no unstable oscillatory solutions (assuming $R_T > R_S > 0$). There are unstable, growing, salt-finger solutions if

$$(n+k)^{\frac{2}{3}} > (n+k)^2. \tag{4.7}$$

Since $0 \leqslant k \leqslant \frac{1}{2}$ this inequality can be satisfied only if $n = 0$ or $n = -1$. If $k$ is small then the unstable solutions to the cubic (4.6) are

(i) $$\lambda = \frac{\tau k^2}{R_T - R_S} + O(k^3) \quad \text{when } n = 0, \tag{4.9}$$

(ii) $$\lambda = \frac{4\tau k}{\dfrac{\tau}{\sigma} + \dfrac{R_S}{\tau} - \sigma R_T} + O(k^2) \quad \text{when } n = -1. \tag{4.9}$$

It can similarly be shown that if $\hat{W}$ and $\mu$ are small, with $\hat{W} = O(\mu)$, then there are two unstable solutions to (3.5): one with

$$\lambda = \frac{\tau\mu^2}{R_T - R_S} + O(\mu^3), \tag{4.10}$$

and the other with $$\lambda = \frac{4\tau k}{\dfrac{\tau}{\sigma} + \dfrac{R_S}{\tau} - \tau R_T} + O(\mu^2). \tag{4.11}$$

We now look at solutions of the truncated set of equations (4.1)–(4.3). We suppose $k/m = O(1)$ and $\mu^2 \ll 1$ and look for a solution $\lambda = O(\mu)$, unlike Holyer (1981), who looked for solutions with $\lambda = O(1)$. We do not need to assume that the Prandtl number $\sigma$ is large. If we eliminate $T_1$ and $S_1$ from (4.1) and $T_{-1}$ and $S_{-1}$ from (4.2) and work to lowest order in $\mu$, we obtain

$$\psi_1\left[\lambda\left(\frac{R_S}{\tau} - \tau R_T + \frac{\tau}{\sigma}\right) + 4k\tau\right] = \frac{m\hat{W}}{2}\left[\tau R_T T_0 - R_S S_0 + \psi_0\left(\frac{\tau}{\sigma} + \frac{R_S}{\tau} - \tau R_T\right)\right], \tag{4.12}$$

$$\psi_{-1}\left[\lambda\left(\frac{R_S}{\tau} - \tau R_T + \frac{\tau}{\sigma}\right) - 4k\tau\right] = \frac{m\hat{W}}{2}\left[\tau R_T T_0 - R_S S_0 - \psi_0\left(\frac{\tau}{\sigma} + \frac{R_S}{\tau} - \tau R_T\right)\right], \tag{4.13}$$

$$T_1 = \tau S_1 = \psi_1, \quad T_{-1} = \tau S_{-1} = -\psi_{-1}. \tag{4.14}$$

Then substituting into (4.3) gives

$$R_T T_0 = R_S S_0, \tag{4.15}$$

$$\tau(\lambda S_0 - k\psi_0) = \lambda T_0 - k\psi_0, \tag{4.16}$$

$$\lambda T_0 - k\psi_0 = \frac{m^2 \hat{W}^2\left(\dfrac{\tau}{\sigma}+\dfrac{R_S}{\tau}-\tau R_T\right)}{\lambda^2\left(\dfrac{\tau}{\sigma}+\dfrac{R_S}{\tau}-\tau R_T\right)^2 - 16k^2\tau^2}[\lambda(\tau R_T T_0 - R_S S_0) - 4k\tau\psi_0]. \tag{4.17}$$

These can be combined to give

$$\lambda^2\left(\frac{\tau}{\sigma}+\frac{R_S}{\tau}-\tau R_T\right)^2 = 16k^2\tau^2 - \frac{m^2\hat{W}^2}{R_T-R_S}\left(\frac{\tau}{\sigma}+\frac{R_S}{\tau}-\tau R_T\right)(4\tau+(1-\tau)^2 R_T(1+R_T)). \tag{4.18}$$

If $\lambda^2 > 0$ then the system is unstable. Hence the system is unstable if

$$m^2\hat{W}^2\left(\frac{\tau}{\sigma}+\frac{R_S}{\tau}-\tau R_T\right)\left(5\tau+(R_T-R_S)\left(\frac{R_S}{\tau}-\tau R_T\right)\right) < 16k^2\tau^2(R_T-R_S). \tag{4.19}$$

If $\hat{W} = 0$ then (4.18) reduces to

$$\lambda^2 = \frac{16k^2\tau^2}{\left(\dfrac{\tau}{\sigma}+\dfrac{R_S}{\tau}-\tau R_T\right)^2}. \tag{4.20}$$

The positive value for $\lambda$ obtained from this is the same as that given by (4.11). This instability, with the condition for instability given by (4.19), shows how the addition of the salt-finger motions modifies the instability that is present when $\hat{W} = 0$. It is the modification of the salt-finger stability problem in an already-present salt-finger field.

If we look at solutions of (4.1)–(4.3) with $\mathrm{i}\omega = O(\mu^2)$ we might expect to obtain (4.10) when $\hat{W}$ is small, and some modification to (4.10) if $\hat{W}$ is larger. If $k/m = O(1)$, $\mu^2 \ll 1$ and $\lambda = O(\mu^2)$ then

$$4k\tau\psi_1 = \frac{m\hat{W}}{2}\left[R_T T_0 - R_S S_0 + \psi_0\left(\frac{\tau}{\sigma}+\frac{R_S}{\tau}-\tau R_T\right)\right], \tag{4.21}$$

$$4k\tau\psi_{-1} = -\frac{m\hat{W}}{2}\left[R_T T_0 - R_S S_0 - \psi_0\left(\frac{\tau}{\sigma}+\frac{R_S}{\tau}-\tau R_T\right)\right], \tag{4.22}$$

$$T_1 = \tau S_1 = \psi_1, \quad T_{-1} = \tau S_{-1} = -\psi_{-1}. \tag{4.23}$$

Substituting into (4.3) gives

$$R_T T_0 = R_S S_0, \tag{4.24}$$

$$(\lambda+\mu^2)T_0 - k\psi_0 = \tau(\lambda+\tau\mu^2)S_0 - \tau k\psi_0, \tag{4.25}$$

$$(\lambda+\mu^2)T_0 - k\psi_0 = \frac{m^2\hat{W}^2}{4k\tau}\psi_0\left(\frac{\tau}{\sigma}+\frac{R_S}{\tau}-\tau R_T\right). \tag{4.26}$$

These give

$$\lambda\left[R_T - R_S - \frac{m^2\hat{W}^2}{4k^2\tau^2}\left(\frac{\tau}{\sigma}+\frac{R_S}{\tau}-\tau R_T\right)\right] = \mu^2\left[\tau + \frac{\lambda m^2\hat{W}^2}{4k^2\tau}\left(\frac{\tau}{\sigma}+\frac{R_S}{\tau}-\tau R_T\right)\right]. \tag{4.27}$$

If $\hat{W} = 0$ then (4.27) reduces to (4.10). The solution (4.27) is unstable if

$$\frac{m^2\hat{W}^2}{4k^2\tau}\left(\frac{\tau}{\sigma}+\frac{R_S}{\tau}-\tau R_T\right) < R_T - R_S. \tag{4.28}$$

The instabilities we have looked at so far have been modifications to the basic salt-finger problem. They provide useful checks for the numerics in §5, but they are already well-understood instabilities. We now, however, find an instability with a new form, by looking for a solution with $k = 0$. Putting $k = 0$ in (4.1) and (4.2) and eliminating $T_1$, $S_1$, $T_{-1}$ and $S_{-1}$, we find

$$(\psi_1+\psi_{-1})P = m\hat{W}(R_T T_0(\lambda+\tau\mu_1^2)-R_S S_0(\lambda+\mu_1^2)), \tag{4.29}$$

$$(\psi_1-\psi_{-1})P = m\hat{W}R\psi_0, \tag{4.30}$$

where
$$P = \mu_1^2\left(\frac{\lambda}{\sigma}+\mu_1^2\right)(\lambda+\tau\mu_1^2)(\lambda+\mu_1^2)+R_T(\lambda+\tau\mu_1^2)-R_S(\lambda+\mu_1^2), \tag{4.31a}$$

$$R = \frac{1}{\sigma}(\lambda+\mu_1^2)(\lambda+\tau\mu_1^2)(1-m^2)-R_T(\lambda+\tau\mu_1^2)+\frac{R_S}{\tau}(\lambda+\mu_1^2), \tag{4.31b}$$

$$\mu_1^2 = 1+m^2. \tag{4.31c}$$

Then substituting into (4.3) gives

$$\left(\frac{\lambda}{\sigma}+m^2\right)\psi_0 = \frac{m^2\hat{W}^2}{2\sigma}\frac{R}{P}\psi_0, \tag{4.32a}$$

$$(\lambda+m^2)T_0 = \frac{-m^2\hat{W}^2}{2P(\lambda+\mu_1^2)}[T_0(P-(\lambda+2+m^2)(\lambda+\tau\mu_1^2)R_T) + S_0 R_S(\lambda+2+m^2)(\lambda+\mu_1^2)] \tag{4.32b}$$

$$(\lambda+\tau m^2)S_0 = \frac{-m^2\hat{W}^2}{2P(\lambda+\tau\mu_1^2)}\left[S_0(P+\left(\frac{\lambda}{\tau}+2+m^2\right)(\lambda+\mu_1^2)R_S) - T_0 R_T\left(\frac{\lambda}{\tau}+2+m^2\right)(\lambda+\tau\mu_1^2)\right]. \tag{4.32c}$$

Using the assumption that $m^2 \ll 1$, then

$$P = \frac{1}{\sigma}(\lambda^3+\lambda^2(\sigma+\tau+1)+\lambda\tau)+\lambda\left(\frac{R_S}{\tau}-\tau R_T\right), \tag{4.33a}$$

$$R = \frac{1}{\sigma}(\lambda^2+\lambda(\sigma+\tau+1)+\tau)+\frac{R_S}{\tau}-\tau R_T, \tag{4.33b}$$

so
$$P = \lambda R. \tag{4.34}$$

We see from (4.32) that there are two different types of solution that occur when $k = 0$:

either (*a*) $\psi_0 = 0$ and $T_0 \neq 0$ and $S_0 \neq 0$,
or (*b*) $T_0 = S_0 = 0$ and $\psi_0 \neq 0$.

For case (*a*) the solutions are stable oscillations, and since they are stable they are not of interest here.

For case (*b*) $T_0 = S_0 = 0$, and (4.32*a*) gives

$$\frac{\lambda}{\sigma}+m^2 = \frac{m^2\hat{W}^2}{2\sigma}\frac{1}{\lambda}. \tag{4.35}$$

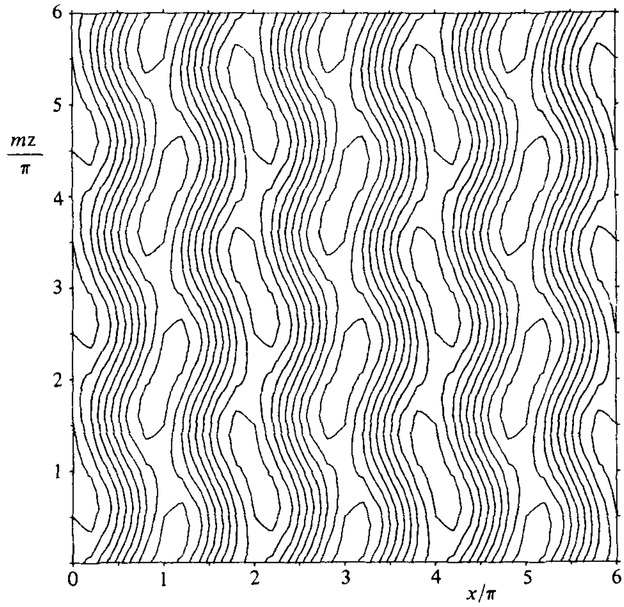


FIGURE 1. Streamlines for the salt fingers and the non-oscillatory perturbation, obtained from the stream function (4.38).

Hence
$$\lambda = -\tfrac{1}{2}\sigma m^2 \pm (\tfrac{1}{2}m^2 \hat{W}^2 + \tfrac{1}{4}\sigma^2 m^4)^{\frac{1}{2}}. \tag{4.36}$$

By selecting the positive sign in (4.36) we have a non-oscillatory solution with a positive growth rate. Note that although we have assumed $m^2 \ll 1$, it is consistent to have terms of order $m$ and order $m^2$ in (4.36), provided that $\sigma m/W = O(1)$. We also find that
$$\psi_1 = -\psi_{-1} = T_1 = T_{-1} = \tau S_1 = \tau S_{-1} = \frac{(\lambda + \sigma m^2)\,\psi_0}{m\hat{W}}. \tag{4.37}$$

This is a new instability. In §5 we find that its growth rate can be larger than that of the collective instability. The stream function for this perturbation is given by
$$\psi = -\hat{W}\cos x - \hat{W}A\sin x \sin mz + \hat{W}B\cos mz, \tag{4.38}$$

where
$$A = \frac{2\psi_1\,\mathrm{e}^{\lambda t}}{\hat{W}}, \quad B = \frac{\psi_0\,\mathrm{e}^{\lambda t}}{\hat{W}}. \tag{4.39}$$

In figure 1 we show the stream function when $A = 0.5$ and $B = 0.14$. These values imply that $\sigma m/\hat{W} = 1.5$. Note that the vertical scale in the figure is compressed, because $m$ is small. Recirculating regions start to grow where the shear in the salt fingers is a maximum, and where the velocity is zero.

## 5. Numerical results

We are now in a position to look at the solutions of (3.5) for the growth rates $\lambda$, and to relate them to the various analytic results found in §4. We display selected numerical results.

First we consider a heat–salt system, where we take $\sigma = 10$ and $\tau = 0.01$. We choose the flux ratio $\gamma$ to be 0.5, which is close to the value found experimentally. If we

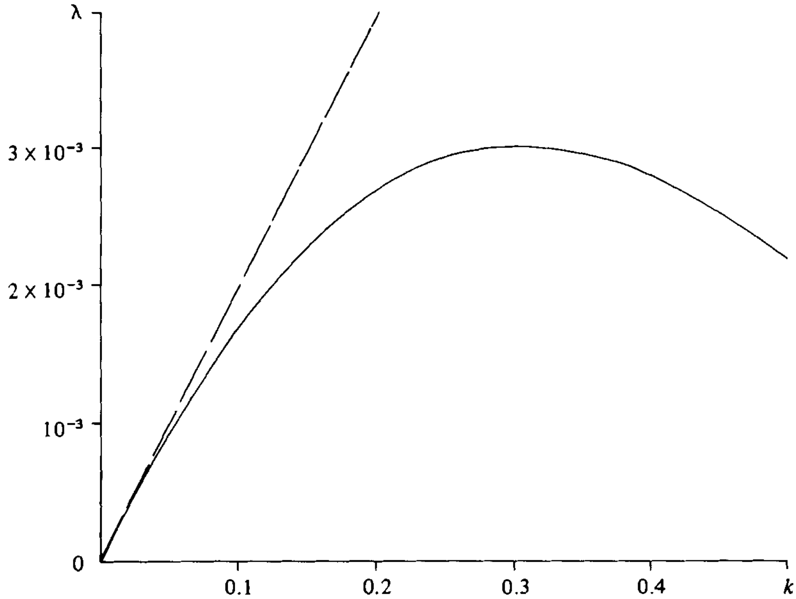


FIGURE 2. Perturbation growth rate plotted against $k$, for $\sigma = 10$, $\tau = 0.01$, $m = 0$ and $\gamma = 0.5$. The solid line is the solution of (3.5). The dashed line is the approximate solution (4.9).

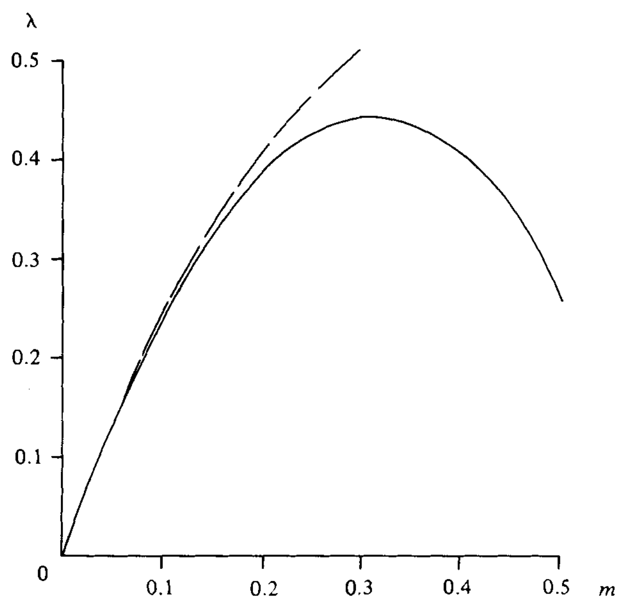


FIGURE 3. Perturbation growth rate plotted against $m$, for $\sigma = 10$, $\tau = 0.01$, $k = 0$, $\gamma = 0.5$ and $\hat{W} = 4$. The solid line is the solution of (3.5). The dashed line is the approximate solution (4.36).

look at perturbations with $m = 0$ (i.e. no vertical variation) then there is no interaction with the salt fingers, and the growth rates are independent of the finger amplitude $\hat{W}$. We show in figure 2 a comparison between the largest growth rate obtained by solving the matrix equation (3.5) with the approximate solution (4.9). This growth rate is independent of $\hat{W}$. The solution displayed in (3.5) is exactly the solution of the cubic equation (4.6) with $n = -1$. This instability is simply salt fingers. Their maximum growth rate is 0.00304, which occurs when $k = 0.30$. As explained at

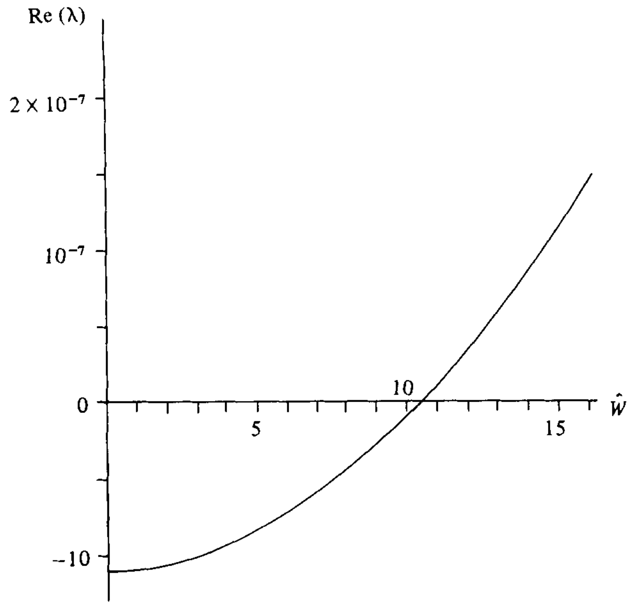


FIGURE 4. Perturbation growth rate for the collective instability plotted against $\hat{W}$, for $\sigma = 10$, $\tau = 0.01$, $k = m = 10^{-4}$ and $\gamma = 0.5$.

the end of §3, we are interested in instabilities with larger growth rates. In figure 3 we show a comparison between the largest growth rate obtained from solving (3.5) with the equation (4.36) for perturbations with $k = 0$. The results are displayed for $\hat{W} = 4$. There is a maximum growth rate of 0.45, which occurs when $m = 0.30$. Figure 4 shows the real part of the growth rate $\lambda$ for the collective instability when $k = m = 10^{-4}$ for the heat–salt system. According to the theory of the collective instability (see (4.4)), in the present case marginal stability should occur when $\hat{W} = 3.61$. In fact we see from figure 4 that it occurs when $\hat{W}$ is about 10.5. The theory of Holyer (1981) assumes that $\sigma \gg 1$ and $\sigma k^2(R_T - R_S)/\mu^2 \gg 1$, as well as $\mu^2 \ll 1$. It appears that for the heat–salt system the first two of these conditions do not hold. If $k$ and $m$ are decreased even further the collective instability still first appears at the same value of $\hat{W}$. From the results that we have looked at, it appears that the growth rate of the collective instability, obtained from the linear stability analysis, is never the largest growth rate present for the heat–salt system. The largest growth rate for the heat–salt system seems to be that from the non-oscillatory instability with $k = 0$. When $\hat{W} = 4$, so that $(\beta F_S - \alpha F_T)/\nu(\alpha T_z - \beta S_z) = 0.82$, the largest growth rate is 0.45, which in dimensional terms is $0.45\,[(1-\gamma)/\sigma\gamma]^{\frac{1}{2}}\,(\alpha g T_z)^{\frac{1}{2}} = 0.1432\,(\alpha g T_z)^{\frac{1}{2}}$. The growth rate of the collective instability is always smaller by at least a factor of 10.

We shall now look at a salt–sugar system, which is used for many laboratory experiments. We take $\sigma = 1000$, $\tau = \frac{1}{3}$, and we choose $\gamma = 0.91$, to be consistent with values obtained experimentally. Figure 5 shows a comparison between the largest growth rate obtained from solving (3.5) with the approximate solution (4.9) when $m = 0$. It is equivalent to figure 2 for the heat–salt system. The maximum growth rate is 0.0187, and occurs when $k = 0.25$. In figure 6 we show a comparison between the largest growth rate obtained from solving (3.5) with the equation (4.36) for perturbations with $k = 0$. The results are displayed for $\hat{W} = 100$, i.e. $(\beta F_S - \alpha F_T)/\nu(\alpha T_z - \beta S_z) = 0.78$. The largest growth rate is 3.89, and occurs at

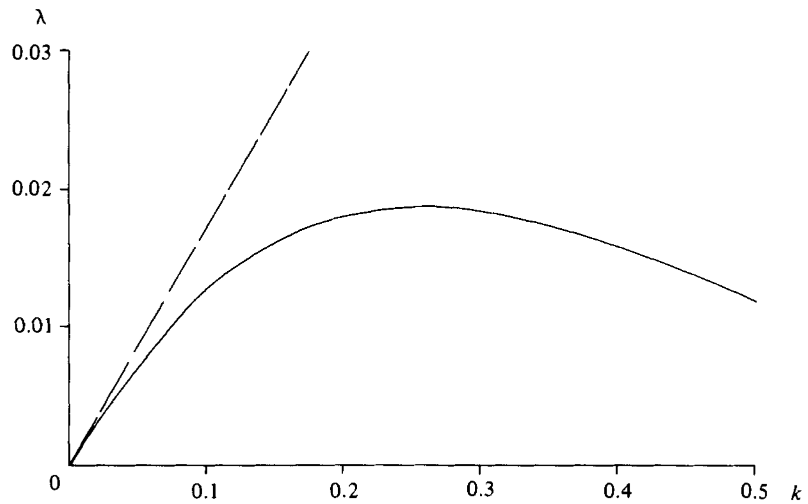


FIGURE 5. Perturbation growth rate plotted against $k$, for $\sigma = 1000$, $\tau = \frac{1}{3}$, $m = 0$ and $\gamma = 0.91$. The solid line is the solution of (3.5). The dashed line is the approximate solution (4.9).

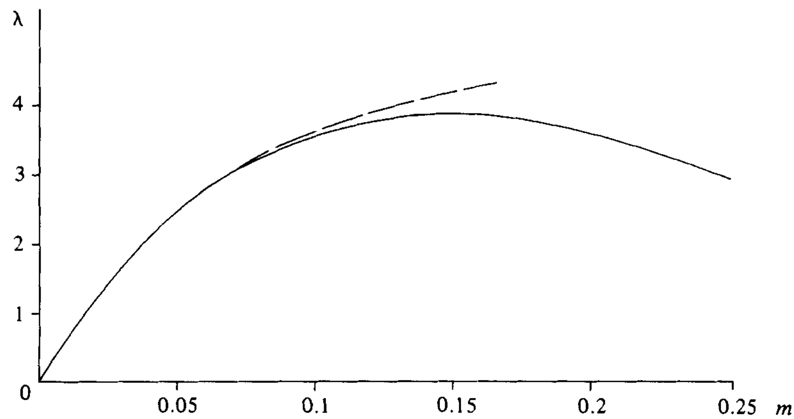


FIGURE 6. Perturbation growth rate plotted against $m$, for $\sigma = 1000$, $\tau = \frac{1}{3}$, $k = 0$, $\gamma = 0.91$ and $\hat{W} = 100$. The solid line is the solution of (3.5). The dashed line is the approximate solution (4.36).

$m = 0.15$. In dimensional terms this is $3.89\ [(1-\gamma)/\sigma\gamma]^{\frac{1}{2}}(\alpha g T_z)^{\frac{1}{2}} = 0.039\,(\alpha g T_z)^{\frac{1}{2}}$. In figure 7 we show the growth rates for the collective instability when $k = m = 0.01$. In this case the theory for the collective instability gives marginal stability for $\hat{W} = 92.43$. We see from figure 7 that the transition happens when $\hat{W} = 95.2$. We have investigated the collective instability for many values of $k$ and $m$, when $\hat{W} = 100$. The largest growth rate for the collective instability is 1.29, and it occurs when $k = 0.05$ and $m = 0.09$. For the salt–sugar system, when $\hat{W} = 100$, the largest growth rate happens when $k = m = 0.5$. At this point $\lambda = 5.8 + 39.8\mathrm{i}$. This instability is probably responsible for the small-scale varicose oscillations that are seen on salt fingers in the laboratory. The frequency of these oscillations is not close to the internal-wave frequency, which is 56.6. The largest growth rate for the non-oscillatory instability, when $\hat{W} = 100$, is 3.89 and occurs when $k = 0$ and $m = 0.15$.

As well as looking at the heat–salt and salt–sugar systems, we have also considered solutions of (3.5) when $\sigma = 10^4$, $\tau = 0.01$ and $\gamma = \frac{1}{3}$. For this case the Prandtl number is sufficiently large that the theory of Holyer (1981) is applicable. In figure 8 we show

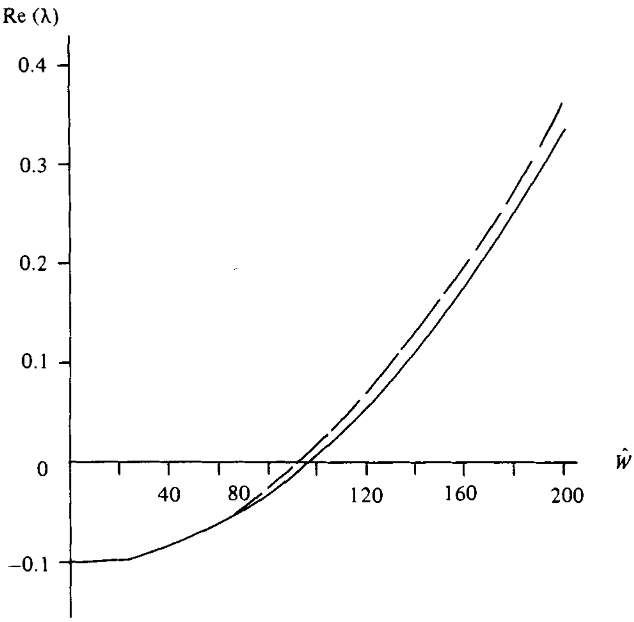


FIGURE 7. Perturbation growth rate for the collective instability plotted against $W$, for $\sigma = 1000$, $\tau = \frac{1}{3}$, $k = m = 0.01$ and $\gamma = 0.91$.

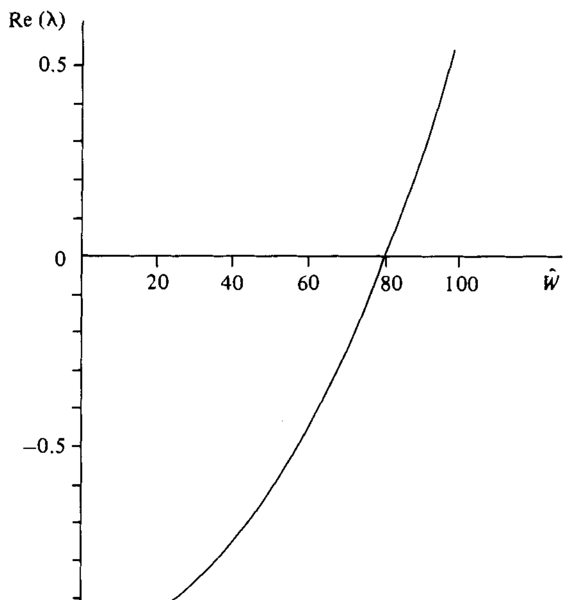


FIGURE 8. Perturbation growth rate plotted against $\hat{W}$, for $\sigma = 10^4$, $\tau = 0.01$, $k = m = 0.01$ and $\gamma = \frac{1}{3}$.

the growth rate for the collective instability when $k = m = 0.01$. The growth rate now agrees closely with the predictions of Holyer (1981). Marginal stability occurs when $\hat{W} = 80.42$. Provided that $\hat{W}$ is larger than this value, then the collective instability has the largest growth rate.

## 6. Conclusions

We have investigated the linear stability of long, steady, two-dimensional salt fingers to two-dimensional pertubations of all wavelengths. Since the basic salt-finger state is spatially periodic we used Floquet theory to determine the growth rate of perturbations. The numerical results we obtained agree with the analytic results of Holyer (1981) for large Prandtl number and long-wavelength perturbations. The addition of perturbations of all wavelengths shows that the collective instability, where the perturbation oscillates at the internal wave frequency, does not have the largest linear growth rate for either the heat–salt system or the salt–sugar system. The collective instability has the largest growth rate only if the Prandtl number is very large. In addition, short-wavelength instabilities appear that may be responsible for the bulges and other small-scale irregularities that can be seen on long salt fingers.

We have found, both analytically and numerically, a new, non-oscillatory instability of salt fingers, which has a larger growth rate than the collective instability for both the heat–salt system and the salt–sugar system. This result is true however large the fluxes are through the salt fingers. This new, non-oscillatory instability requires for its existence the periodic basic state, fluid viscosity and stratification. It starts to grow on the lines where the vertical velocity of the salt-finger field is zero; that is, where the shear is maximum. As it grows to large amplitude it is possible that the recirculating regions of flow will amalgamate and completely disrupt the salt-finger flow. This instability, rather than the collective instability, may limit the length of salt fingers growing at a sharp interface. It could provide an explanation of why Lambert & Demenkow (1972) found salt fingers that were limited in length by fluxes much less than the fluxes predicted by the collective instability theory. More work will be needed to investigate the nonlinear development of this instability, in order to establish whether it is a significant process in double-diffusive motions.

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